R-GROUPS FOR UNITARY PRINCIPAL SERIES OF *Spin* GROUPS

DUBRAVKA BAN, KWANGHO CHOIY, AND DAVID GOLDBERG

ABSTRACT. We study the unitary principal series of the split group $\mathrm{Spin}_m(F)$, where F is a p-adic field. Let $\widetilde{\chi}$ be a unitary character of a maximal F-split torus $\widetilde{T}$ of $\mathrm{GSpin}_m(F)$, and let χ be its restriction to $T = \widetilde{T} \cap \mathrm{Spin}_m(F)$. The R-groups $R_{\widetilde{\chi}}$ and R_χ of the corresponding principal series representations fit in the exact sequence $0 \to R_{\widetilde{\chi}} \to R_\chi \to R_\chi/R_{\widetilde{\chi}} \to 0$. We give a complete answer to the question of splitting of this exact sequence. We also prove that the multiplicity is one when irreducible constituents in unitary principal series are restricted from $\mathrm{GSpin}(F)$ to $\mathrm{Spin}(F)$. Further, based on Keys' result, we prove Arthur's conjecture on R-groups for unitary principal series of Spin.

CONTENTS

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1. Introduction

In a continuation of [10], we study the Knapp-Stein R-group for unitary principal series of split spinor groups over a p-adic field F. The aim of the study is two-fold: to understand the structure of this R-group in terms of the Knapp-Stein R-group of $\mathrm{GSpin}_m(F)$ and to prove multiplicity one in restriction from $\mathrm{GSpin}_m(F)$ to $\mathrm{Spin}_m(F)$ of irreducible constituents in unitary principal series. Furthermore, we prove Arthur's conjecture on R-groups for unitary principal series of Spin, and more generally for those of quasi-split groups, based on Keys' result in [23]. The group Spin_m is the derived group of GSpin_m, and such group structure allows us to look into restriction of tempered representations using theories developed in [18, 33, 20]. The Knapp-Stein R-group of a parabolically induced representation of $\mathrm{GSpin}_m(F)$ is an elementary 2-group [11]; the corresponding R-group of $\mathrm{Spin}_m(F)$ is unknown, except for unitary principal series described by Keys in [22]. In a way that this present paper gives a complete investigation, for the case of principal unitary series, on the splitting of the exact sequence consisting of R-groups of GSpin and Spin, we hope that our approach could shed some light on understanding the unavailable structure of R-groups for any parabolic induction—and hence all tempered non-discrete series representations of $\mathrm{Spin}(F)$—by means of those of $\mathrm{GSpin}(F)$ which are available from [11].

To explain our results more precisely, we let $\mathbf{T}$ be a maximal F-split torus of Spin_m, $m = 2n+1$, or $2n$. Then there is a maximal F-split torus $\widetilde{\mathbf{T}}$ of GSpin such that $\mathbf{T} = \widetilde{\mathbf{T}} \cap \mathrm{Spin}_m$. Set $\widetilde{T} = \widetilde{\mathbf{T}}(F)$ and $T = \mathbf{T}(F)$. Let $\widetilde{\chi}$ be a unitary character of $\widetilde{T}$ and $\chi = \widetilde{\chi}|_T$. Denote by $W(\widetilde{\chi})$ and $W(\chi)$ the Weyl group stabilizers of $\widetilde{\chi}$ and χ in the Weyl group $W = W(\mathbf{G}, \mathbf{T})$. Let $R_{\widetilde{\chi}}$ and R_χ be the Knapp-Stein R-groups of the principal series representations induced from $\widetilde{\chi}$ and χ, respectively. We have the following relationships:

$$W(\widetilde{\chi}) \subseteq W(\chi) \quad \text{and} \quad R_{\widetilde{\chi}} \subseteq R_\chi.$$

Let $(\widetilde{T}/T)^D = \mathrm{Hom}(\widetilde{T}/T, \mathbb{C}^\times)$ and

$$\widehat{W(\chi)} = \{\eta \in (\widetilde{T}/T)^D : {}^w\widetilde{\chi} \simeq \widetilde{\chi}\eta \text{ for some } w \in W(\chi)\}.$$

From [20, Proposition 3.2], where a similar study is done for the case of U_n and SU_n, we have the following exact sequence

$$1 \longrightarrow R_{\widetilde{\chi}} \longrightarrow R_\chi \longrightarrow \widehat{W(\chi)} \longrightarrow 1. \tag{1.1}$$

We investigate whether R_χ is isomophic to the semi-direct product of an 2-elementary group $R_{\widetilde{\chi}}$ and an abelian group $\widehat{W(\chi)}$.

For Spin_{2n+1}, using the description of Knapp-Stein R-group for GSpin in [11] and studying continuous characters of $F^\times$, it follows that $R_{\widetilde{\chi}} = 1$ (Lemma 4.1), and it is obvious that $R_\chi \simeq \widehat{W(\chi)} = \widehat{W(\chi)} \ltimes R_{\widetilde{\chi}}$.

For Spin_{2n}, the investigation becomes more subtle and it turns out to rely on the parity of n. When n is even, the exact sequence (8.7) always splits (Theorem 5.2). In the course of the proof, we verify that in the case when R_χ is non-abelian, the group $R_{\widetilde{\chi}}$ is precisely the subgroup R'_χ of R_χ consisting of all even sign changes (Corollary 5.3). When n is odd, however, the exact sequence (8.7) does not split unless R_χ is an elementary 2-group (see Proposition 5.4 and Proposition 5.6). In the case when R_χ is non-abelian, we know from [22] that $R_\chi \cong \mathbb{Z}_4 \ltimes \mathbb{Z}_2^d$. In this case, we show that $R_{\widetilde{\chi}}$ contains all even sign changes (Lemma 5.1) and is equal to R'_χ, which yields $R_{\widetilde{\chi}} \simeq \mathbb{Z}_2^{d+1}$. In addition, we prove that the exact sequence (8.7) also does not split in the case $R_\chi \cong \mathbb{Z}_4$. Furthermore, we construct an example for Spin_{10} (Section 5.4), using Asgari's description of root systems of GSpin and Spin in [6].

Our other main result is multiplicity one in the restriction of irreducible tempered non-discrete series of $\mathrm{GSpin}(F)$ to $\mathrm{Spin}(F)$ (Theorem 7.2). From our work [10], we know that the multiplicity in restriction of an irreducible tempered representation of $\mathrm{GSpin}(F)$ to $\mathrm{Spin}(F)$ is equal to the multiplicity in restriction of the corresponding irreducible representation of R_χ to $R_{\widetilde{\chi}}$ (see Remark 6.4 for the details). Thus, the proof is reduced to consideration of representations of finite groups. We prove a general result about multiplicity in restrictions of groups of the form $A \rtimes B$, where A and B are finite abelian (Lemma 7.1). This gives the the multiplicity one property in the case when $R_\chi \simeq \widehat{W(\chi)} \ltimes R_{\widetilde{\chi}}$, and also in the case when the semi-direct product does not hold, using the fact that R'_χ equals $R_{\widetilde{\chi}}$. Theorem 7.2 verifies the multiplicity one conjecture by Adler and Prasad (see [1, 2, 16]) for our case of $\mathrm{GSpin}, \mathrm{Spin}$, alongside the case of $\mathrm{GSpin}_4, \mathrm{GSpin}_6$ in [7].

Furthermore, we prove Arthur's conjecture on R-groups for unitary principal series of quasi-split groups over a p-adic field, by revisiting Keys' result on R-groups [23], namely, that the Knapp-Stein, endoscopic, Langlands-Arthur R-groups are isomorphic one another (Theorem 8.2), and we make further remarks for Spin in Section 8.3).

The paper is organized as follows. In Section 2, basic definitions and backgrounds are reviewed. Section 3 describes the group structures of $\mathrm{GSpin}_m(F)$ and $\mathrm{Spin}_m(F)$. Section 4 recalls the structure of the Knapp-Stein R-group of $\mathrm{GSpin}_m(F)$ from [11]. In Section 5, we study the R-group for unitary principal series of $\mathrm{Spin}_m(F)$ in terms of that of $\mathrm{GSpin}_m(F)$. In Section 6, we make an argument on multiplicity in restriction for principal series of connected reductive groups. The multiplicity one in restriction of tempered non-discrete series representations from $\mathrm{GSpin}_m(F)$ to $\mathrm{Spin}_m(F)$ is proved in Section 7.2. Lastly, Arthur's conjecture on R-groups for unitary principal series of quasi-split groups over a p-adic field is proved in Section 8.

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2. Basic notions and backgrounds

Let F be a p-adic field of characteristic 0.

2.1. The R-groups. Let $\mathbf{M}$ be an F-Levi subgroup of a connected reductive algebraic group $\mathbf{G}$ over F. We write $\mathbf{A_M}$ for the split component $\mathbf{A_M}$ in the center of $\mathbf{M}$. We let $\Phi(P, A_M)$ denote the set of reduced roots of $\mathbf{P}$ with respect to $\mathbf{A_M}$. Denote by $W_M = W(\mathbf{G}, \mathbf{A_M}) := N_{\mathbf{G}}(\mathbf{A_M})/Z_{\mathbf{G}}(\mathbf{A_M})$ the Weyl group of $\mathbf{A_M}$ in $\mathbf{G}$, where $N_{\mathbf{G}}(\mathbf{A_M})$ is the normalizer of $\mathbf{A_M}$ in $\mathbf{G}$. We write $\mathrm{Irr}(M)$ for the set of isomorphism classes of irreducible admissible complex representations of M and $\Pi_{\mathrm{disc}}(M)$ for the subset of $\mathrm{Irr}(M)$ consisting of discrete series representations. Given $\sigma \in \mathrm{Irr}(M)$ and $w \in W_M$, we write ${}^w\sigma$ for the representation given by ${}^w\sigma(x) = \sigma(w^{-1}xw)$. Note that the isomorphism class of ${}^w\sigma$ is independent of the choices of representatives in G of $w \in W_M$. Given $\sigma \in \Pi_{\mathrm{disc}}(M)$, we define the stabilizer of σ in W_M

$$W(\sigma) := \{w \in W_M : {}^w\sigma \simeq \sigma\}.$$

Write Δ'_σ for $\{\alpha \in \Phi(P, A_M) : \mu_\alpha(\sigma) = 0\}$, where $\mu_\alpha(\sigma)$ is the rank one Plancherel measure for σ attached to α [19, p.1108]. The *Knapp-Stein R-group* is defined by

$$R_\sigma := \{w \in W(\sigma) : w\alpha > 0, \ \forall \alpha \in \Delta'_\sigma\}.$$

We denote by W°_σ the subgroup of $W(\sigma)$, generated by the reflections in the roots of Δ'_σ. It is well-known from [24, 31, 32] that

$$W(\sigma) = R_\sigma \ltimes W^\circ_\sigma.$$

2.2. Smooth unitary characters of $\mathrm{GL}_1(F)$. Let F be a p-adic field of characteristic 0. Denote by $\mathcal{O}_F$ the ring of integers of F, by $\mathfrak{p}$ the maximal ideal in $\mathcal{O}_F$, and by $\varpi \in F$ a uniformizer of $\mathfrak{p}$. Let q denote the cardinality of the residue field $\mathcal{O}_F/\mathfrak{p}$, and let $|\cdot|_F$ denote the normalized absolute value on F so that $|\varpi|_F = q^{-1}$. Denote by $\mathcal{O}_F^\times$ the group of units in $\mathcal{O}_F$. We have an isomorphism of topological groups:

$$\mathrm{GL}_1(F) = F^\times \simeq \langle\varpi\rangle \times \mathcal{O}_F^\times \simeq \mathbb{Z} \times \mathcal{O}_F^\times,$$

which implies that any $x \in F^\times$ is of the form $\varpi^{v(x)} \cdot u$ with $v(x) \in \mathbb{Z}$ and $u \in \mathcal{O}_F^\times$. Thus, any smooth (quasi-)character χ of $F^\times$ is of the form

$$\chi(x) = \chi(\varpi^{v(x)} \cdot u) = |x|_F^s \cdot \chi^*(u),$$

where $s \in \mathbb{C}$ and χ^* is a character on $\mathcal{O}_F^\times$. Note that since $\mathcal{O}_F^\times$ is compact, χ^* must be unitary, i.e. its image is in the unit circle S^1. Therefore, any smooth character χ of $F^\times$ is unitary if and only if $s \in \sqrt{-1}\mathbb{R}$ (see [29, Sections 7 and 9]). Let us put $s = \sqrt{-1}t$ with $t \in \mathbb{R}$. Then since we have

$$|x|_F^{\sqrt{-1}\cdot t} = q^{-\sqrt{-1}\cdot t\cdot v(x)} = e^{-\sqrt{-1}\cdot t\cdot v(x)\cdot \log q},$$

and since $v(x)$ is an integer, it suffices to consider $0 \le t \lneq \frac{2\pi}{\log q}$. About the unitary character χ^* on $\mathcal{O}_F^\times$, since we have the filtration

$$\mathcal{O}_F^\times \supset 1 + \mathfrak{p} \supset 1 + \mathfrak{p}^2 \supset 1 + \mathfrak{p}^3 \supset \cdots \supset \{1\},$$

χ^* must be a character on a finite group $\mathcal{O}_F^\times/(1+\mathfrak{p}^m) \simeq (\mathcal{O}_F/\mathfrak{p}^m)^\times$ for some $m \in \mathbb{N}$. We note that $\mathcal{O}_F^\times \simeq \mathbb{Z}/(q-1)\mathbb{Z} \times (1+\mathfrak{p})$ (cf. [28, Chapter II.5]). Therefore, any smooth unitary character χ of $F^\times$ is of the form

$$\chi(x) = |x|_F^{\sqrt{-1}t} \cdot \chi^*(u),$$

where $0 \leq t \lneq \frac{2\pi}{\log q}$ and χ^* is a character on a finite group $(\mathcal{O}_F/\mathfrak{p}^m)^\times$ for some $m \in \mathbb{N}$.

Remark 2.1. *Due to the above description, for any character $\chi_1 \otimes \chi_2$ of $F^\times \times F^\times$, we have that $\chi_1 \otimes \chi_2$ is unitary if and only if both χ_1 and χ_2 are unitary characters of $F^\times$.*

3. Structure of Spin in exact sequences

3.1. **Spin groups.** The derived group of GSpin_m, $m = 2n, 2n+1$, is isomorphic to Spin_m. Recall from [8, Proposition 2.2] the following isomorphism of algebraic groups

$$\mathrm{GSpin}_m \simeq (\mathrm{GL}_1 \times \mathrm{Spin}_m)/\{(1,1),(-1,c)\},$$

where c is the nontrivial central element of Spin_m as described in *loc. cit.* (also, see [14, 2.2(5)]). This yields the exact sequence of algebraic groups

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{GL}_1 \times \mathrm{Spin}_m \longrightarrow \mathrm{GSpin}_m \longrightarrow 1.$$

Applying Galois cohomology, we have the exact sequence of F-points

$$1 \longrightarrow \{\pm 1\} \longrightarrow F^\times \times \mathrm{Spin}_m(F) \longrightarrow \mathrm{GSpin}_m(F) \longrightarrow H^1(F, \{\pm 1\}) \longrightarrow 1,$$

which implies that

$$1 \longrightarrow (F^\times \times \mathrm{Spin}_m(F))/\{\pm 1\} \longrightarrow \mathrm{GSpin}_m(F) \longrightarrow F^\times/(F^\times)^2 \longrightarrow 1$$

is exact.

Likewise, the exact sequence of algebraic groups

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}_m \longrightarrow \mathrm{SO}_m \longrightarrow 1 \tag{3.1}$$

gives the exact sequence of F-points

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}_m(F) \longrightarrow \mathrm{SO}_m(F) \longrightarrow H^1(F, \{\pm 1\}) \longrightarrow 1,$$

which implies that

$$1 \longrightarrow \mathrm{Spin}_m(F)/\{\pm 1\} \longrightarrow \mathrm{SO}_m(F) \longrightarrow F^\times/(F^\times)^2 \longrightarrow 1$$

is exact.

Note from [28, Corollary II.5.8] that $|F^\times/(F^\times)^2| = 2 \cdot |\mu_{\mathbf{2}}(F)| \cdot |2|_F^{-1}$ and, as groups, we have

$$F^\times/(F^\times)^2 \simeq \begin{cases} (\mathbb{Z}/2\mathbb{Z})^2, & \text{if } p \neq 2 \\ (\mathbb{Z}/2\mathbb{Z})^r, & \text{if } p = 2, \end{cases} \tag{3.2}$$

where $r \geq 3$.

Using arguments in [14, 2.2(5)] and [27, Section 2.1], a map (surjective homomorphism of algebraic groups) from $\mathrm{GL}_1 \times \mathrm{Spin}_m$ to GL_1 defined by $(z, g) \mapsto z^2$ for any $(z, g) \in \mathrm{GL}_1 \times \mathrm{Spin}_m$ gives a central surjective homomorphism of algebraic groups

$$f : \mathrm{GSpin}_m \longrightarrow \mathrm{GL}_1,$$

since $z \mapsto z^2 : \mathrm{GL}_1 \to \mathrm{GL}_1$ is surjective. It follows that

$$\ker f = (\{\pm 1\} \times \mathrm{Spin}_m)/\{(1, 1), (-1, c)\} \simeq \mathrm{Spin}_m.$$

We then have the following exact sequence of algebraic groups

$$1 \longrightarrow \mathrm{Spin}_m \longrightarrow \mathrm{GSpin}_m \longrightarrow \mathrm{GL}_1 \longrightarrow 1$$

and of complex groups

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \widehat{\mathrm{GSpin}_m} \longrightarrow \widehat{\mathrm{Spin}_m} \longrightarrow 1,$$

where

$$\widehat{\mathrm{GSpin}_{2n}} = \mathrm{GSO}_{2n}(\mathbb{C}),\ \widehat{\mathrm{GSpin}_{2n+1}} = \mathrm{GSp}_{2n}(\mathbb{C}),$$
$$\widehat{\mathrm{Spin}_{2n}} = \mathrm{PSO}_{2n}(\mathbb{C}),\ \widehat{\mathrm{Spin}_{2n+1}} = \mathrm{PSp}_{2n}(\mathbb{C}).$$

3.2. **Levi subgroups and maximal tori.** Let $\theta \subset \Delta$ with the set Δ of simple roots of Spin_m. We identify Δ with the set of simple roots of GSpin_m. Let $M = M_\theta$ be an F-Levi subgroup of Spin_m and let $\widetilde{M} = \widetilde{M}_\theta$ be an F-Levi subgroup of GSpin_m. Let A_θ be the split component in the center of M and let $\widetilde{A}_\theta$ be the split component in the center of $\widetilde{M}$.

The following exact sequence of algebraic groups

$$1 \longrightarrow A_\theta \cap M_{\mathrm{der}} \longrightarrow A_\theta \times M_{\mathrm{der}} \longrightarrow M \longrightarrow 1$$

yields the exact sequence of F-points

$$1 \longrightarrow A_\theta(F) \cap M_{\mathrm{der}}(F) \longrightarrow A_\theta(F) \times M_{\mathrm{der}}(F) \longrightarrow M(F) \longrightarrow H^1(F, A_\theta \cap M_{\mathrm{der}}) \longrightarrow 1.$$

Fix a maximal F-split torus $\widetilde{\mathbf{T}}$ of GSpin_m, and set $\mathbf{T} = \widetilde{\mathbf{T}} \cap \mathrm{Spin}_m$. Then $\mathbf{T}$ is a maximal F-split torus of Spin_m (note that the groups $\mathrm{GSpin}_m, \mathrm{Spin}_m$, under consideration all split over F), which fits into the following exact sequence of algebaic groups

$$1 \longrightarrow \mathbf{T} \longrightarrow \widetilde{\mathbf{T}} \longrightarrow \mathrm{GL}_1 \longrightarrow 1,$$

due to [14, 2.2(5)]. In fact, this exact sequence splits so that we have $\widetilde{\mathbf{T}} \simeq \mathbf{T} \times \mathrm{GL}_1$, where $m = 2n, 2n+1$. We also have another exact sequence

$$1 \longrightarrow T \longrightarrow \widetilde{T} \longrightarrow F^\times \longrightarrow 1,$$

which splits so that $\widetilde{T} \simeq T \times F^\times$. Furthermore, from (3.1), we have

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathbf{T} \longrightarrow \mathbf{T}_0 \longrightarrow 1,$$

where $\mathbf{T}_0 \simeq (\mathrm{GL}_1)^n$ is a maximal torus in SO_m. We may check that $T \simeq (F^\times)^n$, and applying Galois cohomology, we have the following exact sequence of F-points

$$1 \longrightarrow \{\pm 1\} \longrightarrow T \simeq (F^\times)^n \longrightarrow T_0 \simeq (F^\times)^n \longrightarrow H^1(F, \{\pm 1\}) \simeq F^\times/(F^\times)^2 \longrightarrow 1.$$

When $n=1$, we have a particular exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow F^\times \xrightarrow{2} F^\times \longrightarrow F^\times/(F^\times)^2 \longrightarrow 1,$$

where the map $F^\times \xrightarrow{2} F^\times$ means $t \mapsto t^2$.

4. Revisiting Knapp-Stein R-group for unitary principal series of GSpin

Let $\widetilde{\mathbf{T}}$ be a maximal F-split torus of GSpin_m, $m = 2n+1$, or $2n$. Then, we have

$$\widetilde{\mathbf{T}} \simeq (\mathrm{GL}_1)^{n+1}$$

(c.f., [8, proof of Proposition 2.3]). Any unitary character $\widetilde{\chi}$ of $\widetilde{T}$ is of the form

$$\widetilde{\chi}_1 \otimes \widetilde{\chi}_2 \otimes \cdots \widetilde{\chi}_n \otimes \tau,$$

where $\widetilde{\chi}_i$ and τ are all unitary characters of $F^\times$ (cf. Remark 2.1).

By convention, we have $\mathrm{GSpin}_0 = \mathrm{GSpin}_1 = \mathrm{GL}_1$, and we note from [8, 9] that

$$\mathrm{GSpin}_2 = \mathrm{GL}_1 \times \mathrm{GL}_1 \text{ and } \mathrm{GSpin}_3 = \mathrm{GL}_2.$$

4.1. **Case** $m = 2n+1$**.** From Keys's result in [22] we have the following description of R-groups R for unitary principal series of Spin_{2n+1}:

Type B_n**.**

Any element in R is conjugate to

$$1,\ (1\ 2)(3\ 4)\cdots(n-1\ n),\ (1\ 2)(3\ 4)\cdots(k-1\ k)C_{k+1}C_{k+2}\cdots C_n,\ C_1C_2\cdots C_n$$

with sign change C_i on e_i, and

$$R \simeq (\mathbb{Z}_2)^d$$

with $d = n - k/2$, where $|R|$ divides both $2n$ and $[F^\times : (F^\times)^2]$.

Next, we use [11, Theorem 2.5] to describe the R-group for unitary principal series of $\mathrm{GSpin}_{2n+1}(F)$.

Lemma 4.1. *Let* $\widetilde{\mathbf{T}}$ *be a maximal* F*-split torus of* GSpin_{2n+1} *and let* $\widetilde{\chi}$ *be a unitary character of* T*. Then we have* $R_{\widetilde{\chi}} = 1$.

Proof. We let

$$\Omega(\widetilde{\chi}) = \{1 \le i \le n : i_{\mathrm{GL}_1\times\mathrm{GL}_1}^{\mathrm{GSpin}_3}(\widetilde{\chi}_i \otimes \tau) \text{ is reducible, } \widetilde{\chi}_i \not\simeq \widetilde{\chi}_j \text{ for all } j > i\}.$$

Since $\mathrm{GSpin}_3 = \mathrm{GL}_2$ and since $i_{\mathrm{GL}_1\times\mathrm{GL}_1}^{\mathrm{GSpin}_3}(\widetilde{\chi}_i \otimes \tau)$ is reducible if and only if $\widetilde{\chi}_i = \tau|\cdot|^{\pm 1}$ (cf. [13]), we have $\Omega(\widetilde{\chi}) = \emptyset$. By [11, Theorem 2.5], we have $R_{\widetilde{\chi}} = 1$. □

4.2. **Case $m = 2n$.** From Keys's result in [22] we have the following description of R-groups for unitary principal series of $Spin_{2n}$:

Type D_n.

(a) Suppose n is even. Then if R is abelian, $R \simeq \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, with the number of factors bounded by $n-1$ and by $[F^\times : (F^\times)^2] - 1$. If R is nonabelian,

$$R \simeq (\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2) \ltimes (\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2) \tag{4.1}$$

with the order of the first factor dividing both $2n$ and $[F^\times : (F^\times)^2]$ and the number of factors of $\mathbb{Z}_2$ in the normal subgroup an odd number bounded by $n-1$ and by $[F^\times : (F^\times)^2] - 1$.

(b) Suppose n is odd. Then if R is abelian, $R \simeq \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, with the number of factors bounded by $n-1$ and by $[F^\times : (F^\times)^2] - 1$, or $R \simeq \mathbb{Z}_4$. If R is nonabelian,

$$R \simeq \mathbb{Z}_4 \ltimes (\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2) \tag{4.2}$$

with the number of factors of $\mathbb{Z}_2$ in the normal subgroup an even number bounded by $n-3$ and by $[F^\times : (F^\times)^2] - 2$.

Next, we use [11, Theorem 2.7] to describe the R-group for unitary principal series of $\mathrm{GSpin}_{2n}(F)$. Let $\widetilde{T}$ be a maximal F-split torus in $\mathrm{Spin}_{2n}(F)$ and $\widetilde{T}$ the corresponding maximal F-split torus in $\mathrm{GSpin}_{2n}(F)$. Let

$$\widetilde{\chi} = \widetilde{\chi}_1 \otimes \cdots \otimes \widetilde{\chi}_n \otimes \tau$$

be a unitary character of $\widetilde{T} \simeq T \times F^\times$. Let

$$\Omega_2(\widetilde{\chi}) = \{1 \le i \le n \mid \widetilde{\chi}_i \simeq \widetilde{\chi}_i^{-1} \otimes \tau \text{ and } \widetilde{\chi}_j \not\simeq \widetilde{\chi}_i \text{ for all } j > i\}. \tag{4.3}$$

Set $d_2 = |\Omega_2(\widetilde{\chi})|$. Then, from [11, Theorem 2.7], we have

$$R_{\widetilde{\chi}} = \langle C_i C_j : i, j \in \Omega_2(\widetilde{\chi}) \rangle \simeq \mathbb{Z}_2^{d_2 - 1}. \tag{4.4}$$

Proposition 4.2. *Suppose $d_2 \neq 0$. Then, d_2 is bounded by n and by $[F^\times : (F^\times)^2]$.*

Proof. It is clear from the definition that d_2 is bounded by n. Since $d_2 \neq 0$, we let i_0 be the smallest integer in $\Omega_2(\widetilde{\chi})$. Suppose $\widetilde{\chi}_j \in \Omega_2(\widetilde{\chi})$. Then, we have

$$\widetilde{\chi}_{i_0}^2 = \tau = \widetilde{\chi}_j^2 \ \text{ and } \ \widetilde{\chi}_{i_0} \neq \widetilde{\chi}_j.$$

The first condition yields $(\widetilde{\chi}_{i_0} \widetilde{\chi}_j^{-1})^2 = 1$. It follows that

$$\widetilde{\chi}_j = \widetilde{\chi}_{i_0} \cdot \mathsf{sgn}_\theta,$$

where $\bar{\theta} \in F^\times/(F^\times)^2$, which implies that the number of such $\widetilde{\chi}_j$ is bounded by $[F^\times : (F^\times)^2] - 1$. Therefore, the number of elements in Ω_2 is at most $[F^\times : (F^\times)^2]$. □

5. Knapp-Stein R-group for unitary principal series of Spin

As before, fix a maximal F-split torus $\widetilde{\mathbf{T}}$ of GSpin_m, and set $\mathbf{T} = \widetilde{\mathbf{T}} \cap \mathrm{Spin}_m$. Let $\widetilde{\chi}$ be a unitary character of $\widetilde{T}$ and $\chi = \widetilde{\chi}|_T$. Then $W(\widetilde{\chi}) \subseteq W(\chi)$ and $R_{\widetilde{\chi}} \subseteq R_\chi$. Following [20, Proposition 3.2], since $X(\widetilde{\chi}) = \{\eta \in (\widetilde{T}/T)^D : \widetilde{\chi}\eta \simeq \widetilde{\chi}\} = 1$, we have the following exact sequence

$$1 \longrightarrow R_{\widetilde{\chi}} \longrightarrow R_\chi \longrightarrow \widehat{W(\chi)} \longrightarrow 1, \tag{5.1}$$

where $\widehat{W(\chi)} = \{\eta \in (\widetilde{T}/T)^D : {}^w\widetilde{\chi} \simeq \widetilde{\chi}\eta \text{ for some } w \in W(\chi)\}$.

5.1. **Case** Spin_{2n+1}**.** We know from Lemma 4.1 that in this case $R_{\widetilde{\chi}} = 1$. Then it follows trivially that the sequence (5.1) splits, and

$$R_\chi \simeq \widehat{W(\chi)} = \widehat{W(\chi)} \ltimes R_{\widetilde{\chi}}.$$

5.2. **Case** Spin_{2n}**.** We want to know if the sequence (5.1) splits for Spin_{2n}. It turns out that the answer depends on whether n is even or odd.

We start with the following simple observation.

Lemma 5.1. *Let S be any subset of $R_{\widetilde{\chi}}$ and let $\mathcal{C}$ be the set of all sign changes appearing in any product in S. Then $R_{\widetilde{\chi}}$ contains all even sign changes built from the elements of $\mathcal{C}$.*

Proof. This follows directly from [11, Theorem 2.7]. □

Let R'_χ be the subgroup of R_χ consisting of even sign changes. From (4.4) and the definition of R'_χ, it is clear that

$$R_{\widetilde{\chi}} \subset R'_\chi. \tag{5.2}$$

5.3. **Case** Spin_{2n}**,** n **even.**

Theorem 5.2. *Let $\mathbf{G} = \mathrm{Spin}_{2n}$ and $\widetilde{\mathbf{G}} = \mathrm{GSpin}_{2n}$, with n even. Then the exact sequence*

$$1 \longrightarrow R_{\widetilde{\chi}} \longrightarrow R_\chi \longrightarrow \widehat{W(\chi)} \longrightarrow 1$$

splits, so $R_\chi \simeq \widehat{W(\chi)} \ltimes R_{\widetilde{\chi}}$.

Proof. Case 1: R_χ abelian. Due to (5.1) and an argument in [22, p.371], if R_χ has no elements of order 4, then $R_\chi \simeq (\mathbb{Z}/2\mathbb{Z})^r$ for some integer r and the exact sequence (5.1) always splits. Thus, we have

$$R_\chi \simeq \widehat{W(\chi)} \ltimes R_{\widetilde{\chi}}.$$

Case 2: R_χ non-abelian. From now on, we assume that R_χ has an element of order 4.

It follows from the paragraph above Lemma 2 and the proof of Lemma 2 in [22] that we have the following exact sequence

$$1 \longrightarrow R'_\chi \longrightarrow R_\chi \longrightarrow R_\chi/R'_\chi \longrightarrow 1, \tag{5.3}$$

and the factor group R_χ/R'_χ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$ for some integer r. Furthermore, due to Lemma 4 in *op. cit.*, the exact sequence (5.3) splits and we have

$$R_\chi \simeq R_\chi/R'_\chi \ltimes R'_\chi.$$

We use now a more explicit description of R_χ from [22, Case 1, p.371]. Up to conjugation by an element of W, R_χ has the following form. There exist ℓ odd and m even such that $1 \le \ell \le m-3 \le m \le n$ and R'_χ is the group of all even sign changes on $\{e_\ell, \ldots, e_m\}$. In addition, R_χ contains

$$x = (1\,2)(3\,4)\cdots(m-1\,m)C_{m+1}\cdots C_n.$$

Let r be an odd number satisfying $\ell \le r \le m-3$. Set $a = C_r C_{r+2}$. Then

$$xax^{-1}a^{-1} = C_r C_{r+1} C_{r+2} C_{r+3}.$$

Note that $R_{\widetilde{\chi}}$ contains the commutator subgroup of R_χ, because $R_\chi/R_{\widetilde{\chi}} \cong \widehat{W(\chi)}$ is abelian. It follows $C_r C_{r+1} C_{r+2} C_{r+3} \in R_{\widetilde{\chi}}$. Lemma 5.1 tells us that $R_{\widetilde{\chi}}$ contains all even sign changes on $\{e_r, e_{r+1}, e_{r+2}, e_{r+3}\}$. Since r is arbitrary, it follows that $R_{\widetilde{\chi}}$ contains all even sign changes on $\{e_\ell, \ldots, e_m\}$, that is, $R_{\widetilde{\chi}} = R'_\chi$. $\square$

From the proof of the previous theorem, we have

Corollary 5.3. *Let* $\mathbf{G} = \mathrm{Spin}_{2n}$ *and* $\widetilde{\mathbf{G}} = \mathrm{GSpin}_{2n}$, *with* n *even. If the* R*-group* R_χ *is non-abelian, then* $R_{\widetilde{\chi}} = R'_\chi$, *the subgroup of* R_χ *consisting of even sign changes.*

5.4. **An example in** Spin_{10}**.** The example below illustrates the process of restriction from $\widetilde{T}$ to T, which is nontrivial. It also gives an example of a non-split exact sequence (5.1).

The set of simple roots of Spin_{10} is $\{\alpha_1, \cdots, \alpha_5\}$ and the set of simple coroots is $\{\alpha_1^\vee, \cdots, \alpha_5^\vee\}$, where

$$\alpha_i = \begin{cases} f_i - f_{i+1}, & i < 5, \\ f_4 + f_5, & i = 5, \end{cases} \quad \text{and} \quad \alpha_i^\vee = \begin{cases} f_i^* - f_{i+1}^*, & i < 5, \\ f_4^* + f_5^*, & i = 5. \end{cases}$$

Let

$$T = \{\alpha_1^\vee(s_1)\cdots\alpha_5^\vee(s_5) \mid s_i \in F^\times\} \cong (F^\times)^5$$

be our maximal F-split torus in Spin_{10}. The isomorphism $T \cong (F^\times)^5$ comes from the fact that Spin_n is simply connected.

Let λ be a character of T. By identifying T with $(F^\times)^5$, we can write

$$\lambda = \lambda_1 \otimes \lambda_2 \otimes \cdots \otimes \lambda_5,$$

where $\lambda_i = \lambda \circ \alpha_i^\vee$, $i = 1, \ldots, 5$. Let $\lambda_1 = \mathsf{sgn}_\theta$, $\lambda_2 = \mathsf{sgn}_{\theta'}$ be two different characters of $F^\times$ of order two, and let Ψ be a character of $F^\times$ of order four such that $\Psi^2 = \lambda_1$. Take

$$\lambda = \lambda_1 \otimes \lambda_2 \otimes \lambda_1 \otimes \Psi \otimes \Psi^{-1} = \mathsf{sgn}_\theta \otimes \mathsf{sgn}_{\theta'} \otimes \mathsf{sgn}_\theta \otimes \Psi \otimes \Psi^{-1}.$$

Let $\Delta' = \{\alpha > 0 \mid \lambda_\alpha \equiv 1\}$ and

$$R = R_\lambda = \{w \in W(\lambda) \mid w(\Delta') = \Delta'\}.$$

It can be shown by direct computation that in this example $\Delta' = \emptyset$, so $R_\lambda = W(\lambda)$. From page 373 in [22], the case $m = k = 4 = n-1$, we have the element

$$x = (1\,2)(3\,4)C_4C_5 \in R.$$

In addition, we can show by direct computation that

$$C_1C_2 \in W_\lambda, \quad C_1C_3 \in W_\lambda, \quad C_1C_4 \in W_\lambda, \quad C_1C_5 \notin W_\lambda.$$

We conclude that R', the group of sign changes in R, consists of all even sign changes on $\{1,2,3,4\}$. We can write R as $R = \langle x \rangle \ltimes R_0$, where $R_0 = \langle C_1C_3, C_2C_4 \rangle$. Alternatively, we can take $R_0 = \langle C_1C_4, C_2C_3 \rangle$.

Now, as in [6], set

$$e_i^* = f_i^* + (f_0^*)/2,\ 1 \le i \le n, \quad e_0^* = f_0^*.$$

Then the simple coroots of GSpin_{10} are

$$\alpha_i^\vee = \begin{cases} e_i^* - e_{i+1}^*, & i < 5, \\ e_4^* + e_5^* - e_0^*, & i = 5. \end{cases}$$

We want to find a character $\widetilde{\chi}$ of $\widetilde{T}$ such that $\widetilde{\chi}|_T = \lambda$. Write $\widetilde{\chi} = \widetilde{\chi}_0 \otimes \widetilde{\chi}_1 \otimes \cdots \otimes \widetilde{\chi}_5$. Then for $t = \prod_{i=0}^5 e_i^*(t_i)$ we have $\widetilde{\chi}(t) = \prod_{i=0}^5 \widetilde{\chi}_i(t_i)$. If

$$t = \alpha_1^\vee(s_1)\cdots\alpha_5^\vee(s_5) = \prod_{i=0}^5 e_i^*(t_i) \in T,$$

then

$$\begin{aligned} t_0 &= s_5^{-1} & t_3 &= s_2^{-1}s_3 \\ t_1 &= s_1 & t_4 &= s_3^{-1}s_4s_5 \\ t_2 &= s_1^{-1}s_2 & t_5 &= s_4^{-1}s_5 \end{aligned}$$

We have

$$\begin{aligned} \prod_{i=0}^5 \widetilde{\chi}_i(t_i) &= \widetilde{\chi}_0(s_5^{-1})\widetilde{\chi}_1(s_1)\widetilde{\chi}_2(s_1^{-1}s_2)\widetilde{\chi}_3(s_2^{-1}s_3)\widetilde{\chi}_4(s_3^{-1}s_4s_5)\widetilde{\chi}_5(s_4^{-1}s_5) \\ &= \lambda_1(s_1)\lambda_2(s_2)\lambda_1(s_3)\Psi(s_4)\Psi^{-1}(s_5). \end{aligned}$$

It follows

$$\begin{aligned} \widetilde{\chi}_1\widetilde{\chi}_2^{-1} &= \lambda_1 & \widetilde{\chi}_4\widetilde{\chi}_5^{-1} &= \Psi \\ \widetilde{\chi}_2\widetilde{\chi}_3^{-1} &= \lambda_2 & \widetilde{\chi}_0^{-1}\widetilde{\chi}_4\widetilde{\chi}_5 &= \Psi^{-1} \\ \widetilde{\chi}_3\widetilde{\chi}_4^{-1} &= \lambda_1 \end{aligned} \tag{5.4}$$

Take an arbitrary character φ of $F^\times$ and define

$$\begin{aligned} \widetilde{\chi}_0 &= \varphi^2 & \widetilde{\chi}_3 &= \varphi\lambda_1 \\ \widetilde{\chi}_1 &= \varphi\lambda_2 & \widetilde{\chi}_4 &= \varphi \\ \widetilde{\chi}_2 &= \varphi\lambda_1\lambda_2 & \widetilde{\chi}_5 &= \varphi\Psi^{-1} \end{aligned}$$

Then $\widetilde{\chi}_0, \ldots, \widetilde{\chi}_5$ satisfy equation (5.4). Hence,

$$\widetilde{\chi} = \varphi^2 \otimes \varphi\lambda_2 \otimes \varphi\lambda_1\lambda_2 \otimes \varphi\lambda_1 \otimes \varphi \otimes \varphi\Psi^{-1}$$

satisfies $\widetilde{\chi}|_T = \lambda = \lambda_1 \otimes \lambda_2 \otimes \lambda_1 \otimes \Psi \otimes \Psi^{-1}$.

The R-group $R_{\widetilde{\chi}}$ can be found using [11, Theorem 2.7]. In the notation of [11], $\tau = \omega_\tau = \widetilde{\chi}_0 = \varphi^2$. Note that

$$\widetilde{\chi}_1^{-1}\widetilde{\chi}_0 = \varphi^{-1}\lambda_2^{-1}\varphi^2 = \varphi\lambda_2 = \widetilde{\chi}_1.$$

Similarly, $\widetilde{\chi}_i^{-1}\widetilde{\chi}_0 = \widetilde{\chi}_i$ for $i = 1, 2, 3, 4$. For $i = 5$, we get

$$\widetilde{\chi}_5^{-1}\widetilde{\chi}_0 = \varphi^{-1}\Psi\varphi^2 = \varphi\Psi \neq \widetilde{\chi}_5.$$

Then [11, Theorem 2.7] tells us that $R_{\widetilde{\chi}} \cong \mathbb{Z}_2^3$ and $R_{\widetilde{\chi}} = \langle C_iC_j \mid 1 \leq i < j \leq 4\rangle$. Alternatively, we can find $R_{\widetilde{\chi}}$ using the equality $R_{\widetilde{\chi}} = R_\lambda \cap W(\widetilde{\chi})$. It follows

$$R_{\widetilde{\chi}} = R' \cong \mathbb{Z}_2^3.$$

Note that $R/R' \cong \mathbb{Z}_2$. The group R has 8 elements of order four. It follows that the sequence

$$1 \to R_{\widetilde{\chi}} \to R \to R/R_{\widetilde{\chi}} \to 1$$

does not split.

5.5. **Case** Spin_{2n}**,** n **odd.** Let $\widetilde{\chi}$ be a unitary character of $\widetilde{T}$ and $\chi = \widetilde{\chi}|_T$. As we saw from the example in Section 5.4, there are cases when the exact sequence (5.1) does not split. If $R_\chi \simeq \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, then it splits trivially. For the other case of abelian R_χ, we have the following.

Proposition 5.4. *If* $R_\chi \simeq \mathbb{Z}_4$, *then* $R_{\widetilde{\chi}} \simeq \mathbb{Z}_2$ *and thus* (5.1) *does not split.*

Proof. We recall from [22, p.374] that $R_\chi = \langle (1\,2)(3\,4)\cdots(m-1\,m)C_mC_{m+1}\cdots C_n\rangle$. Write x for the generator; η for the image of x via the homomorphism $R_\chi \to \widehat{W(\chi)}$ in (5.1); and η' for the image of $x^2 = C_{m-1}C_m$. Since we have, as characters on $\widetilde{T}$,

$$^{x}\widetilde{\chi} \simeq \eta\widetilde{\chi}, \quad {}^{x^2}\widetilde{\chi} \simeq \eta'\widetilde{\chi}, \quad \text{and} \quad {}^{x^2}\widetilde{\chi} \simeq {}^{x}({}^{x}\widetilde{\chi}) \simeq {}^{x}\eta\eta\widetilde{\chi}, \tag{5.5}$$

it follows that

$$^{x}\eta \simeq \eta, \quad \text{and} \quad \eta^2 \simeq \eta'. \tag{5.6}$$

Recall the decomposition $\widetilde{\chi} = \widetilde{\chi}_1 \otimes \cdots \otimes \widetilde{\chi}_n \otimes \tau$, and the actions of $(i\ j)$ and C_k on $\widetilde{\chi}$ for the case GSpin_{2n} with odd n from [11, p.7]. Using ${}^{x}\eta \simeq \eta$ in (5.6) and ${}^{x^2}\widetilde{\chi} \simeq \eta'\widetilde{\chi}$ in (5.5), we have the following decomposition of η and η' :

$$\eta \simeq \eta_1 \otimes \eta_2 \otimes \eta_3 \otimes \eta_4 \otimes \cdots \otimes \eta_{m-1} \otimes \eta_m \otimes \eta_{m+1} \otimes \cdots \otimes \eta_n \otimes \delta$$

with $\eta_1 = \eta_2,\ \eta_3 = \eta_4,\ \ldots,\ \eta_{m-1} = \eta_m,\ \delta = \eta_m^2 = \eta_{m+1}^2 = \cdots = \eta_n^2$; and

$$\eta' \simeq \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \eta'_{m-1} \otimes \eta'_m \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$

with $\eta'_{m-1} = \widetilde{\chi}_{m-1}^{-2}\tau,\ \eta'_m = \widetilde{\chi}_m^{-2}\tau$. Using $\eta^2 \simeq \eta'$ in (5.6), we have

$$\delta^2 = \eta_{m+1}^2 = \eta_{m+2}^2 = \cdots = \eta_n^2 = \mathbb{1},$$

which implies $\eta_{m-1}^2 = 1 = \eta_m^2$. It thus follows that

$$\eta^2 = \mathbb{1} = \eta'.$$

Therefore, the kernel $R_{\widetilde{\chi}}$, which is an elementary 2-group, of the homomorphism $R_\chi \to \widehat{W(\chi)}$ in (5.1) must contain $x^2 = C_{m-1}C_m$, and the proof is complete. □

The rest of this section is devoted to the last case that R_χ is not abelian. We know from [22] that

$$R_\chi \cong \mathbb{Z}_4 \ltimes \mathbb{Z}_2^d, \quad d \text{ even}, \, d \geq 2.$$

The cases $d \geq 4$ are possible only for $p = 2$.

We will need a more explicit description R_χ. There is a small error in [22], which we correct here. Up to conjugation by an element of W, R_χ has the following form. It contains the aforementioned element

$$x = sc = (1\,2)(3\,4)\cdots(m-1\,m)C_mC_{m+1}\cdots C_n \tag{5.7}$$

and the subgroup

$$A = \langle C_{k-1}C_k,\, C_kC_{k+1},\, \ldots,\, C_{m-2}C_{m-1}\rangle.$$

Note that A is not normal in R_χ (as stated in loc.cit.) because

$$xC_{m-2}C_{m-1}x^{-1} = C_{m-3}C_m \notin A.$$

Since $x^2 = C_{m-1}C_m$, we see that R_χ contains all even sign changes on $e_{k-1}, \cdots, e_m$. Let

$$B = \langle x,\, C_{k-1}C_k,\, C_kC_{k+1},\, \ldots,\, C_{m-2}C_{m-1}\rangle.$$

We want to express B as a semidirect product. We follow the approach from the example in Section 5.4. For every $i \in \{k-1, k, \ldots, m-2\}$ define

$$a_i = \begin{cases} C_iC_{m-1}, & i \text{ odd}, \\ C_iC_m, & i \text{ even}. \end{cases} \tag{5.8}$$

Since for any odd $i < m$

$$xC_ix^{-1} = C_{i+1}, \quad xC_{i+1}x^{-1} = C_i,$$

it follows $xa_ix^{-1} = a_{i+1}$ for i odd and

$$B = \langle x\rangle \ltimes (\langle a_{k-1}\rangle \times \cdots \times \langle a_{m-2}\rangle).$$

If there are other sign changes in R_χ, we may assume that, after conjugation by an element of W, they involve $\{e_\ell, e_{\ell+1}, \ldots, e_{k-2}\}$ for some odd $\ell \leq k-1$. Then the group R'_χ of even sign changes in R_χ consists of all even sign changes on $\{e_\ell, e_{\ell+1}, \ldots, e_m\}$. For $i \in \{\ell, \ldots, k-2\}$ define a_i by formula (5.8). Then

$$R_\chi = \langle x\rangle \ltimes (\langle a_\ell\rangle \times \cdots \times \langle a_{m-2}\rangle).$$

Lemma 5.5. *Suppose $R_\chi \cong \mathbb{Z}_4 \ltimes \mathbb{Z}_2^d$ is nonabelian. Then the group R'_χ of even sign changes in R_χ is isomorphic to $\mathbb{Z}_2^{d+1}$ and the exact sequence*

$$1 \longrightarrow R'_\chi \longrightarrow R_\chi \longrightarrow \mathbb{Z}_2 \longrightarrow 1 \tag{5.9}$$

does not split.

Proof. We may assume that R_χ is of the form described above. The isomorphism $R_\chi \simeq \mathbb{Z}_2^{d+1}$ follows from the assumption. Then we can write R_χ as the disjoint union

$$R_\chi = R'_\chi \cup xR'_\chi.$$

Note that all elements of xR'_χ are of order four. It then follows that the sequence (5.9) does not split. □

Proposition 5.6. *If $R_\chi \cong \mathbb{Z}_4 \ltimes \mathbb{Z}_2^d$ is nonabelian, then $R_{\widetilde{\chi}}$ is equal to the group R'_χ of even sign changes in R_χ. In particular, $R_{\widetilde{\chi}} \cong \mathbb{Z}_2^{d+1}$ and the exact sequence*

$$1 \longrightarrow R_{\widetilde{\chi}} \longrightarrow R_\chi \longrightarrow \widehat{W(\chi)} \longrightarrow 1$$

does not split.

Proof. We begin by observing that $R_{\widetilde{\chi}}$ contains the commutator subgroup of R_χ, because $R_\chi/R_{\widetilde{\chi}} \cong \widehat{W(\chi)}$ is abelian.

We may assume that R_χ is of the form described above. For $i \in \{\ell, \dots, m-3\}$ odd and a_i defined by formula (5.8), we have

$$a_i x a_i x^{-1} = C_i C_{i+1} C_{m-1} C_m \in R_{\widetilde{\chi}}.$$

Lemma 5.1 tells us that $R_{\widetilde{\chi}}$ contains all even sign changes on $\{e_i, e_{i+1}, e_{m-1}, e_m\}$. Since i is arbitrary, it follows that $R_{\widetilde{\chi}}$ contains all even sign changes on $\{e_\ell, e_{\ell+1}, \dots, e_m\}$. Hence, $R_{\widetilde{\chi}} = R'_\chi$. □

6. R-groups and multiplicity in restriction in a general setting

We make some arguments about relationships between R-groups and multiplicity in restriction in a general setting, extending a previous work [17, Section 5] for restriction from inner forms of GL_n to those SL_n. Let $\mathbf{G}$ and $\widetilde{\mathbf{G}}$ denote connected reductive groups over F, such that

$$\mathbf{G}_{\mathrm{der}} = \widetilde{\mathbf{G}}_{\mathrm{der}} \subseteq \mathbf{G} \subseteq \widetilde{\mathbf{G}}, \tag{6.1}$$

where $\mathbf{G}_{\mathrm{der}}$ and $\widetilde{\mathbf{G}}_{\mathrm{der}}$ denote the derived groups of $\mathbf{G}$ and $\widetilde{\mathbf{G}}$, respectively. Let $\mathbf{M}$ be an F-Levi subgroup of $\mathbf{G}$, and let $\widetilde{\mathbf{M}}$ be an F-Levi subgroup of $\widetilde{\mathbf{G}}$ such that $\widetilde{\mathbf{M}} = \mathbf{M} \cap \mathbf{G}$. Let $\sigma \in \Pi_{\mathrm{disc}}(M)$ be given. Choose $\widetilde{\sigma} \in \mathrm{Irr}(\widetilde{M})$ such that $\sigma \hookrightarrow \widetilde{\sigma}|_M$. Since restriction and parabolic induction are compatible (cf. [12, Proposition 4.1]), we have the following isomorphism of G-modules

$$(i_{\widetilde{M}}^{\widetilde{G}}(\widetilde{\sigma}))|_G \simeq i_M^G(\widetilde{\sigma}|_M). \tag{6.2}$$

Let $\mathrm{JH}(\star)$ denote the set of all irreducible inequivalent constituents in $i_M^G(\star)$. Given $\pi \in \mathrm{JH}(\sigma)$, we fix a lifting $\widetilde{\pi} \in \mathrm{JH}(\widetilde{\sigma})$. In a similar way to [17, Section 5], counting the multiplicities in both sides in the isomorphism (6.2), we have

$$\begin{aligned} &\mathrm{multi}(\pi, \widetilde{\pi}|_G) \cdot \mathrm{multi}(\widetilde{\pi}, \mathrm{JH}(\widetilde{\sigma})) \cdot \#\Big(\{\eta \in (\widetilde{G}/G)^D : \widetilde{\pi}\eta \in \mathrm{JH}(\widetilde{\sigma})\}/X(\widetilde{\pi})\Big) \\ &\quad = \mathrm{multi}(\pi, \mathrm{JH}(\sigma)) \cdot \#\{\tau \hookrightarrow \widetilde{\sigma}|_M : \pi \hookrightarrow i_M^G(\tau)\}, \end{aligned} \tag{6.3}$$

where $X(\widetilde{\pi}) = \{\eta \in (\widetilde{G}/G)^D : \widetilde{\pi}\eta \in \mathrm{JH}(\widetilde{\chi})\}$.

Remark 6.1. $\{\eta \in (\widetilde{G}/G)^D : \widetilde{\pi}\eta \in \mathrm{JH}(\widetilde{\sigma})\} = \{\eta \in (\widetilde{M}/M)^D : \widetilde{\sigma}\eta \simeq {}^w\widetilde{\sigma}$ *for some* $w \in W_{\widetilde{M}}\}$. *Indeed, since we have* $\widetilde{\pi}\eta \in \mathrm{JH}(\widetilde{\sigma}\eta)$ *considering* η *as a character on* $\widetilde{M}/M$ *via an isomorphism* $\widetilde{G}/G \simeq \widetilde{M}/M$ *(cf.* [15, Proposition 3.2.3]*), it follows that* $\mathrm{JH}(\widetilde{\sigma}\eta) \cap \mathrm{JH}(\widetilde{\sigma}) \neq \emptyset$, *which implies from* [4, Proposition 1.1] *that* $\widetilde{\sigma}\eta \simeq {}^w\widetilde{\sigma}$ *for some* $w \in W_{\widetilde{M}}$. *Conversely, the condition* $\widetilde{\sigma}\eta \simeq {}^w\widetilde{\sigma}$ *yields* $i_{\widetilde{M}}^{\widetilde{G}}(\widetilde{\sigma}) \simeq i_{\widetilde{M}}^{\widetilde{G}}(\widetilde{\sigma}\eta) \simeq i_{\widetilde{M}}^{\widetilde{G}}(\widetilde{\sigma})\eta$, *which implies* $\widetilde{\pi}\eta \in \mathrm{JH}(\widetilde{\sigma})$.

Remark 6.2. (a) *Identifying* $W_{\widetilde{M}} = W_M$, *given* $w \in W(\sigma)$, *we have* $\widetilde{\sigma}\eta \simeq {}^w\widetilde{\sigma}$ *for some* $\eta \in (\widetilde{M}/M)^D$, *which gives the exact sequence*

$$1 \longrightarrow W(\widetilde{\sigma}) \cap W(\sigma) \longrightarrow W(\sigma) \longrightarrow \{\eta \in (\widetilde{M}/M)^D : \widetilde{\sigma}\eta \simeq {}^w\widetilde{\sigma} \text{ for some } w \in W_{\widetilde{M}}\}/X(\widetilde{\sigma}).$$

The cokernel has the following property

$$W(\sigma)/(W(\widetilde{\sigma}) \cap W(\sigma)) \simeq \widehat{W(\sigma)}/X(\widetilde{\sigma}) \hookrightarrow \{\eta \in (\widetilde{M}/M)^D : \widetilde{\sigma}\eta \simeq {}^w\widetilde{\sigma} \text{ for some } w \in W_{\widetilde{M}}\}/X(\widetilde{\sigma}), \tag{6.4}$$

where $\widehat{W(\sigma)} := \{\eta \in (\widetilde{M}/M)^D : \widetilde{\sigma}\eta \simeq {}^w\widetilde{\sigma}$ *for some* $w \in W(\sigma)\}$. *See also* [15, Proposition 3.4.5] *and its proof.*

(b) *From a 2-cocycle in* $H^2(R_\sigma, \mathbb{C}^\times)$, *we have a central extension*

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \widetilde{R_\sigma} \longrightarrow R_\sigma \longrightarrow 1, \tag{6.5}$$

and set $\Pi_-(\widetilde{R_\sigma}) = \{$*irreducible representations* ρ *of* $\widetilde{R_\sigma} : \rho(z) = z \cdot id$ *for all* $z \in \mathbb{C}^\times\}$. *Then, from* [15, Corollary 2.3.3], *we have*

$$i_M^G(\sigma) \simeq \bigoplus_{\rho \in \Pi_-(\widetilde{R_\sigma})} \rho \boxtimes \pi_\rho \tag{6.6}$$

as representations of $\widetilde{R_\sigma} \times G$, *where* $\rho \mapsto \pi_\rho$ *is a bijection from* $\Pi_-(\widetilde{R_\sigma})$ *to* $\mathrm{JH}(\sigma)$. *This decomposition yields*

$$\mathrm{multi}(\pi_\rho, i_M^G(\sigma)) = \dim \rho.$$

Furthermore, from [23, Theorem 2.3], *for any unitary character* σ *of* M, *the central extension* (6.5) *always splits, so that we can identify the set* $\Pi_-(\widetilde{R_\sigma})$ *with that of irreducible representations of* R_σ.

□

Now we consider the case of the minimal F-Levi subgroups of any connected reductive F-group and $\widetilde{\sigma}, \sigma$ are unitary characters $\widetilde{\chi}, \chi$. Note that $\widetilde{\chi}|_M = \chi$. It then follows that $\#\{\tau \hookrightarrow \widetilde{\sigma}|_M : \pi \hookrightarrow i_M^G(\tau)\} = 1$. It is also clear that $W(\widetilde{\chi}) \subset W(\chi)$, i.e., $W(\widetilde{\chi}) \cap W(\chi) = W(\widetilde{\chi})$, and $X(\widetilde{\chi}) = 1$. As discovered in [20, Proposition 3.2], we have the exact sequence

$$1 \longrightarrow R_{\widetilde{\chi}} \longrightarrow R_\chi \longrightarrow \widehat{W(\chi)} \longrightarrow 1. \tag{6.7}$$

Thus, setting $\pi = \pi_\rho, \widetilde{\pi} = \widetilde{\pi}_{\widetilde{\rho}}$ with $\rho \in \mathrm{Irr}(R_\chi), \widetilde{\rho} \in \mathrm{Irr}(R_{\widetilde{\chi}})$ corresponding to $\pi, \widetilde{\pi}$, respectively, the equality (6.3) immediately yields

$$\mathrm{multi}(\pi, \widetilde{\pi}|_G) \cdot \dim \widetilde{\rho} \cdot \#\Big(\{\eta \in (\widetilde{G}/G)^D : \widetilde{\pi}\eta \in \mathrm{JH}(\widetilde{\chi})\}/X(\widetilde{\pi})\Big) = \dim \rho, \tag{6.8}$$

where $\mathrm{Irr}(R_\chi)$ denotes the set of isomorphism classes of irreducible complex representations of R_χ.

Theorem 6.3. *With above notation, we have*

$$\text{multi}(\pi, \widetilde{\pi}|_G) \cdot \dim \widetilde{\rho} \cdot \frac{\#\widehat{W(\chi)}}{\#X(\widetilde{\pi})} = \dim \rho. \tag{6.9}$$

Proof. The condition $\widetilde{\pi}\eta \in \text{JH}(\widetilde{\chi})$ for some $\eta \in (\widetilde{G}/G)^D$ amounts to $\widetilde{\chi}\eta \simeq {}^w\widetilde{\chi}$ for some $w \in W_{\widetilde{M}}$ (cf. [4, Proposition 1.1]), so that we have

$$\begin{aligned} &\{\eta \in (\widetilde{M}/M)^D : \widetilde{\chi}\eta \simeq {}^w\widetilde{\chi} \text{ for some } w \in W_{\widetilde{M}}\} \\ = &\{\eta \in (\widetilde{M}/M)^D : \widetilde{\chi}\eta \simeq {}^w\widetilde{\chi} \text{ for some } w \in W(\chi)\}. \end{aligned}$$

This equality comes from the fact $\widetilde{\chi}\eta \simeq {}^w\widetilde{\chi} \Rightarrow \chi \simeq {}^w\chi$. Thus, (6.8) yields the theorem. □

Remark 6.4. *Consider when $R_{\widetilde{\chi}}$ is abelian (this is the case of $\widetilde{\mathbf{G}} = \text{GSpin}_m$ due to* [11]*; see also* [10]*). We recall and use Ito's result in the representation theory of finite groups (cf.* [21, Theorem 6.15]*): Let two finite groups A, B with $A \lhd B$ and abelian A be given. For any irreducible representation ρ of B, we have* $\dim \rho \mid [B : A]$. *Then, the quotient $\frac{\#\widehat{W(\chi)}}{\dim \rho}$ is an integer. Then, Theorem 6.3 yields*

$$\text{multi}(\pi, \widetilde{\pi}|_G) \cdot \frac{\#\widehat{W(\chi)}}{\dim \rho} = \#X(\widetilde{\pi}) = \#\Pi_{\widetilde{\pi}}(G) \cdot \text{multi}(\pi, \widetilde{\pi}|_G)^2,$$

where $\Pi_{\widetilde{\pi}}(G)$ denotes the set of equivalence classes of all irreducible constituents of $\widetilde{\pi}|_G$. The equality $\#X(\widetilde{\pi}) = \#\Pi_{\widetilde{\pi}}(G) \cdot \text{multi}(\pi, \widetilde{\pi}|_G)^2$ is due to [16, Proposition 3.1]*. Thus, we have*

$$\text{multi}(\pi, \widetilde{\pi}|_G) = \frac{\#\widehat{W(\chi)}}{\dim \rho \cdot \#\Pi_{\widetilde{\pi}}(G)}, \tag{6.10}$$

and furthermore, the exact sequence (6.7) *yields*

$$\text{multi}(\pi, \widetilde{\pi}|_G) = \frac{\#R_\chi}{\#R_{\widetilde{\chi}} \cdot \dim \rho \cdot \#\Pi_{\widetilde{\pi}}(G)}.$$

Remark 6.5. *In particular, when $\widetilde{\mathbf{G}} = \text{GSpin}_m, \mathbf{G} = \text{Spin}_m$, we set $\pi = \pi_\rho$ and $\widetilde{\pi} = \widetilde{\pi}_{\widetilde{\rho}}$ with $\rho \in \text{Irr}(R_\chi)$ and $\widetilde{\rho} \in \text{Irr}(R_{\widetilde{\chi}})$ corresponding to π and $\widetilde{\pi}$, using the decomposition* (6.6) *for* GSpin *and* Spin, *respectively. Then,* [10, Theorem 3.4] *states that*

$$\text{multi}(\pi, \widetilde{\pi}|_G) = \text{multi}(\widetilde{\rho}, \rho|_{R_{\widetilde{\chi}}}). \tag{6.11}$$

We refer the reader to [10] *for the details. In fact, it turns out that the proof works for any principal series of general connected reductive groups $\widetilde{\mathbf{G}}, \mathbf{G}$ satisfying the property* (6.1)*.*

7. Multiplicity one in restriction of unitary principal series from GSpin to Spin

In this section, we prove that the multiplicity in restriction of an irreducible constituent in unitary principal series of $\widetilde{\mathbf{G}} = \text{GSpin}$ to $\mathbf{G} = \text{Spin}$ is always one. This supports the multiplicity one conjecture by Adler and Prasad [1]. For the proof, we need the following lemma on representations of finite groups.

Lemma 7.1. *Let R be a finite group. Suppose $R = A \rtimes B$, where A is normal and A and B are both abelian. Let χ be a character of A and ρ an irreducible representation of R. Then*

$$\text{multi}(\chi, \rho|_A) \leq 1$$

Proof. Assume $\mathrm{multi}(\chi, \rho|_A) > 0$. We apply [30, Proposition 25] which describes all irreducible representations of R. We fix χ as a representative of its orbit in $\mathrm{Hom}(A, \mathbb{C}^\times)$ under the action of B. Let

$$H = \{h \in B \mid h\chi = \chi\}.$$

Note that any irreducible representation of H is a character, because H is abelian. Proposition 25 of [30] tells us that there exists a unique character ψ of H such that

$$\rho \cong i^R_{A \rtimes H}(\chi \otimes \psi).$$

By [30, Proposition 22], we have

$$\begin{aligned} \rho|_{A \rtimes H} &= \mathrm{Res}^R_{A \rtimes H} i^R_{A \rtimes H}(\chi \otimes \psi) \\ &= \bigoplus_{b \in B/H} b(\chi \otimes \psi) = \bigoplus_{b \in B/H} (b\chi) \otimes \psi, \end{aligned}$$

because B acts trivially on ψ. It follows

$$\rho|_A = \bigoplus_{b \in B/H} b\chi.$$

By the construction of H, we have $b\chi \neq \chi$ for $b \in B \setminus H$. It follows $\mathrm{multi}(\chi, \rho|_A) = 1$. □

Let $\widetilde{\mathbf{T}}$ be a maximal F-split torus of GSpin_m, and set $\mathbf{T} = \widetilde{\mathbf{T}} \cap \mathrm{Spin}_m$, which forms a maximal F-split torus of Spin_m. Given a unitary character $\widetilde{\chi}$ of $\widetilde{T}$, we let χ be the restriction $\widetilde{\chi}|_T$ of $\widetilde{\chi}$ to T. Recalling the notation in Section 6, we let $\pi \in \mathrm{JH}(\chi)$, we fix a lifting $\widetilde{\pi} \in \mathrm{JH}(\widetilde{\chi})$.

Theorem 7.2. *With above notation, we have*

$$\mathrm{multi}(\pi, \widetilde{\pi}|_G) = 1.$$

Proof. In the case when $R_\chi \simeq \widehat{W(\chi)} \ltimes R_{\widetilde{\chi}}$, the theorem follows immediately from Lemma 7.1 and Remark 6.5. It remains to consider the case when the exact sequence

$$1 \longrightarrow R_{\widetilde{\chi}} \longrightarrow R_\chi \longrightarrow \widehat{W(\chi)} \longrightarrow 1$$

does not split. From Section 5, this happens for $R_\chi \cong \mathbb{Z}_4 \ltimes \mathbb{Z}_2^d$. In that case, $R_{\widetilde{\chi}}$ is equal to the group R'_χ of even sign changes in R_χ. Note that

$$R_\chi \cong \langle x \rangle \ltimes R''_\chi,$$

where $R''_\chi < R'_\chi = R_{\widetilde{\chi}}$ is the subset consisting of the even sign changes on $\{e_1, e_2, ..., e_{m-1}\}$ (see [22, page 373]) and x is the generator described in (5.7). Let ρ be an irreducible representation of R_χ. Let ρ' be an irreducible component of $\rho|_{R'_\chi}$ and $\rho'' = \rho'|_{R''_\chi}$. Lemma 7.1 tells us that $\mathrm{multi}(\rho'', \rho|_{R''_\chi}) = 1$. Since $\rho'' = \rho'|_{R''_\chi}$, we have

$$\mathrm{multi}(\rho', \rho|_{R'_\chi}) = 1.$$

The theorem follows from Remark 6.5. □

8. Arthur's conjecture on R-groups for unitary principal series for Spin

Based on the result by Keys in [23, Section 2], we prove Arthur's conjecture on R-groups for Spin. We recall notation from previous sections along with the following.

8.1. **Arthur's conjecture on R-groups in a general setting.** Let F be a p-adic field of characteristic 0, W_F the Weil group, and Γ the absolute Galois group $\mathrm{Gal}(\bar{F}/F)$.

Let $\mathbf{G}$ be a connected reductive algebraic group over F. We write $G := \mathbf{G}(F)$ for the group of F-points, and a semi-direct product ${}^LG := \widehat{G} \rtimes W_F$ for the L-group of G fixing Γ-invariant splitting data (see [14, Section 2]). Denote by $\Pi_{\mathrm{temp}}(G)$ the subsets of $\mathrm{Irr}(G)$ which consist of tempered representations. We denote by $\Phi(G)$ the set of equivalence classes of L-parameters for G, where an L-parameter for G stands for an admissible homomorphism $\phi : W_F \times SL_2(\mathbb{C}) \to {}^LG$, and two L-parameters are said to be equivalent if they are conjugate by $\widehat{G}$. We denote by $C_\phi(\widehat{G})$ the centralizer of the image of ϕ in $\widehat{G}$. The center of ${}^LG$ is the Γ-invariant group $Z(\widehat{G})^\Gamma$. Note that $C_\phi \supset Z(\widehat{G})^\Gamma$. We say that ϕ is elliptic if the quotient group

$$C_\phi(\widehat{G})/Z(\widehat{G})^\Gamma$$

is finite, and ϕ is tempered if $\phi(W_F)$ is bounded. We denote by $\Phi_{\mathrm{ell}}(G)$ and $\Phi_{\mathrm{temp}}(G)$ the subset of $\Phi(G)$ which respectively consist of elliptic and tempered L-parameters of G. We set $\Phi_{\mathrm{disc}}(G) = \Phi_{\mathrm{ell}}(G) \cap \Phi_{\mathrm{temp}}(G)$.

The local Langlands conjecture for G predicts that there is a surjective finite-to-one map from $\mathrm{Irr}(G)$ to $\Phi(G)$. Given $\phi \in \Phi(G)$, we write $\Pi_\phi(G)$ for the L-packet attached to ϕ, and then the local Langlands conjecture implies that

$$\mathrm{Irr}(G) = \bigsqcup_{\phi \in \Phi(G)} \Pi_\phi(G).$$

It is expected that $\Phi_{\mathrm{disc}}(G)$ and $\Phi_{\mathrm{temp}}(G)$ respectively parameterize $\Pi_{\mathrm{disc}}(G)$ and $\Pi_{\mathrm{temp}}(G)$.

Given an L-parameter $\phi \in \Phi(M)$, we also consider ϕ as an L-parameter for G via the inclusion $\widehat{M} \hookrightarrow \widehat{G}$. Fix a maximal torus T_ϕ in $C_\phi(\widehat{G})^\circ$. We set

$$W_\phi^\circ := N_{C_\phi(\widehat{G})^\circ}(T_\phi)/Z_{C_\phi(\widehat{G})^\circ}(T_\phi), \quad W_\phi := N_{C_\phi(\widehat{G})}(T_\phi)/Z_{C_\phi(\widehat{G})}(T_\phi).$$

The *endoscopic R-group* R_ϕ is defined as follows

$$R_\phi := W_\phi/W_\phi^\circ.$$

We identify W_ϕ with a subgroup of W_M (see [3, p.45]). For $\sigma \in \Pi_\phi(M)$, we set

$$W_{\phi,\sigma}^\circ := W_\phi^\circ \cap W(\sigma), \quad W_{\phi,\sigma} := W_\phi \cap W(\sigma).$$

The *Langlands-Arthur R-group* $R_{\phi,\sigma}$ is defined as follows

$$R_{\phi,\sigma} := W_{\phi,\sigma}/W_{\phi,\sigma}^\circ.$$

Arthur's conjecture, predicted in [3], states that given a discrete series representation $\sigma \in \Pi_\phi(M)$, we have

$$R_\sigma \simeq R_{\phi,\sigma}.$$

We note from [3, p.46] that

$$R_{\phi,\sigma} \subset R_\phi. \tag{8.1}$$

8.2. **Arthur's conjecture for unitary principal series for general quasi-split groups.** Let $\mathbf{G}$ be a connected reductive quasi-split algebraic group over F. Given a maximal torus $\mathbf{T}$ of $\mathbf{G}$ and a unitary character χ of T and its corresponding L-parameter $\phi \in \Phi_{\mathrm{disc}}(T)$ via the local Langlands correspondence for tori $\mathrm{Hom}(T, \mathbb{C}^\times) \simeq \Phi(T)$, cf. [26, Theorem 1] and [34, Theorem, p.179]), we recall the quotient group $C_\phi(\widehat{G})/Z(\widehat{G})^\Gamma$ from Section 8.1, and denote by $\mathcal{S}_\phi(\widehat{G})$ the group of connected components

$$\pi_0(C_\phi(\widehat{G})/Z(\widehat{G})^\Gamma).$$

For unitary principal series of Spin_m, due to the following exact sequence (see [3, (7.1)] or [5, (4.2.3)])

$$1 \longrightarrow \mathcal{S}_\phi(\widehat{T}) = 1 \longrightarrow \mathcal{S}_\phi(\widehat{G}) \longrightarrow R_\phi \longrightarrow 1,$$

we have

$$R_\phi \simeq \mathcal{S}_\phi(\widehat{G}). \tag{8.2}$$

We recall Keys' results on unitary principal series of quasi-split groups from [23, Section 2]. Due to [23, Lemma 2.5 (ii)], we note that in his notation U, S_ϕ are respectively identified with our T_ϕ, C_ϕ, and his $W(\mathcal{S}_\phi^\circ, U)$ equals our W_ϕ°. Thus, it follows from [23, Proposition 2.6] and (8.1) that

$$W_\phi^\circ = W^\circ(\chi) = W_{\phi,\chi}^\circ. \tag{8.3}$$

Further, due to [23, Proposition 2.6], we have $\mathcal{S}_\phi(\widehat{G}) \simeq R_\chi$, and thus the isomorphism (8.2) implies that

$$R_\chi \simeq \mathcal{S}_\phi(\widehat{G}) \simeq R_\phi. \tag{8.4}$$

Lemma 8.1. *We have*

$$W(\chi) \subset W_\phi.$$

Proof. We let $w \in W_T$ be given such that ${}^w\chi \simeq \chi$. Identifying W_ϕ with a subgroup of W_M (see [3, p.45]) in Section 8.1, due to the local Langlands correspondence for tori ([26, Theorem 1], [34, Theorem, p.179]), we have ${}^w\phi \simeq \phi$. □

Theorem 8.2.

$$R_\chi \simeq \mathcal{S}_\phi(\widehat{G}) \simeq R_\phi \simeq R_{\phi,\chi}.$$

Proof. Due to Lemma 8.1, we have

$$W(\chi) = W_{\phi,\chi} = W_\phi \cap W(\chi).$$

Further, the equalities (8.3) yield

$$R_\chi \simeq R_{\phi,\chi}.$$

Therefore, by (8.4), we have the theorem. □

8.3. **Further remarks and the case of $\widetilde{\mathbf{G}} = \mathrm{GSpin}_m$ and $\mathbf{G} = \mathrm{Spin}_m$.** We let $\widetilde{\mathbf{G}}$ be a connected reductive group over F, and let $\mathbf{G}$ be a closed F-subgroup of $\widetilde{\mathbf{G}}$ such that

$$\mathbf{G}_{\mathrm{der}} = \widetilde{\mathbf{G}}_{\mathrm{der}} \subseteq \mathbf{G} \subseteq \widetilde{\mathbf{G}}, \tag{8.5}$$

where the subscript der stands for the derived group. Given $\phi \in \Phi(G)$, from [25, Théorèm 8.1], there exists a lifting $\widetilde{\phi} \in \Phi(\widetilde{G})$ such that $\phi = pr \circ \widetilde{\phi}$, where pr is the projection $\widehat{\widetilde{G}} \twoheadrightarrow \widehat{G}$. Since the homomorphism pr is compatible with Γ-actions on $\widehat{\widetilde{G}}$ and $\widehat{G}$, the lifting $\widetilde{\phi} \in \Phi(\widetilde{G})$ is chosen uniquely up to a 1-cocycle of W_F in $\widehat{(\widetilde{G}/G)}$ due to [25, Section 7].

Given a maximal F-split torus $\widetilde{\mathbf{T}}$ of $\widetilde{\mathbf{G}}$, we set $\mathbf{T} = \widetilde{\mathbf{T}} \cap \mathbf{G}$, which is a maximal F-split torus of $\mathbf{G}$. Let $\widetilde{\chi}$ be a unitary character of $\widetilde{T}$ and $\chi = \widetilde{\chi}|_T$. By the local Langlands correspondence for tori ([26, Theorem 1], [34, Theorem, p.179]), we denote by $\widetilde{\phi}$ the L-parameter for $\widetilde{\chi}$, and accordingly $\phi = pr \circ \widetilde{\phi}$ is the L-parameter for χ. Recall

$$\widehat{W(\chi)} = \{\eta \in (\widetilde{T}/T)^D : {}^w\widetilde{\chi} \simeq \widetilde{\chi}\eta \text{ for some } w \in W(\chi)\},$$

and set

$$X^{\widetilde{G}}(\widetilde{\phi}) := \{\mathbf{a} \in H^1(W_F, \widehat{\mathrm{GL}_1}) : \mathbf{a}\widetilde{\phi} \simeq \widetilde{\phi}\}.$$

Due to [15, Lemma 5.3.4], we then have the exact sequence

$$1 \to \mathcal{S}_{\widetilde{\phi}}(\widehat{\widetilde{G}}) \to \mathcal{S}_{\phi}(\widehat{G}) \to X^{\widetilde{G}}(\widetilde{\phi}) \to 1,$$

which implies that

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R_{\widetilde{\phi}} \simeq R_{\widetilde{\chi}} \simeq R_{\widetilde{\phi},\widetilde{\chi}} & \longrightarrow & R_{\phi} \simeq R_{\chi} \simeq R_{\phi,\chi} & \longrightarrow & \widehat{W(\chi)} & \longrightarrow & 1 \\ & & \simeq\uparrow & & \simeq\uparrow & & \simeq\uparrow & & \\ 1 & \longrightarrow & \mathcal{S}_{\widetilde{\phi}}(\widehat{\widetilde{G}}) & \longrightarrow & \mathcal{S}_{\phi}(\widehat{G}) & \longrightarrow & X^{\widetilde{G}}(\widetilde{\phi}) & \longrightarrow & 1 \end{array} \tag{8.6}$$

commutes.

Finally, we shall focus on the case of $\widetilde{\mathbf{G}} = \mathrm{GSpin}_m$ and $\mathbf{G} = \mathrm{Spin}_m$. When $\widetilde{\mathbf{G}} = \mathrm{GSpin}_{2n+1}$ and $\mathbf{G} = \mathrm{Spin}_{2n+1}$, we recall $R_{\widetilde{\chi}} = 1$ (Lemma 4.1) and the exact sequence (5.1)

$$1 \longrightarrow R_{\widetilde{\chi}} \longrightarrow R_{\chi} \longrightarrow \widehat{W(\chi)} \longrightarrow 1. \tag{8.7}$$

Due to (8.6), we then have

$$R_{\chi} \simeq \widehat{W(\chi)} \simeq R_{\phi} \simeq R_{\phi,\chi} \simeq \mathcal{S}_{\phi}(\widehat{G}) \simeq X^{\widetilde{G}}(\widetilde{\phi}).$$

Furthermore, when $\widetilde{\mathbf{G}} = \mathrm{GSpin}_{2n}$ and $\mathbf{G} = \mathrm{Spin}_{2n}$, with even n, Section 5.3 implies that two horizontal exact sequences in (8.6) split. For the case of odd n, Section 5.5 gives the full information about the splitness of those two exact sequences in (8.6).

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